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Dynamics on Leibniz manifolds

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Abstract

This paper shows that various well-known dynamical systems can be described as vector fields associated to smooth functions via a bracket that defines what we call a Leibniz structure. We show that gradient flows, some control and dissipative systems, and non-holonomically constrained simple mechanical systems, among other dynamical behaviors, can be described using this mathematical construction that generalizes the standard Poisson bracket currently used in Hamiltonian mechanics. The symmetries of these systems and the associated reduction procedures are described in detail. A number of examples illustrate the theoretical developments in the paper.

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1. Introduction

It is well known that most of classical mechanics can be formulated using a Poisson structure (see for instance [1,19] and references therein). A Poisson structure on a manifold P is a bilinear map $\{\cdot,\cdot\} : C^\infty(P) \times C^\infty(P) \to C^\infty(P)$ that defines a Lie algebra structure on the algebra $C^\infty(P)$ of smooth functions on that manifold and that is a derivation on each entry. This property allows the association of a vector field X_h to any smooth function $h \in C^\infty(P)$, usually referred to as the Hamiltonian vector field of the Hamiltonian function h. The properties of the bracket $\{\cdot,\cdot\}$ have important consequences on the dynamical features

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of the vector field X_h. For instance, its antisymmetry implies that the Hamiltonian function h is a constant of the motion for X_h. Additionally, the fact that $\{\cdot, \cdot\}$ satisfies the Jacobi identity implies that the flow of X_h is a Poisson map, that is, it respects the bracket.

It has been noticed in recent times that a weakening of the defining conditions of a Poisson system is sometimes necessary in order to accommodate the description of more general dynamical systems. A well known example is the use of brackets that do not satisfy the Jacobi identity, known as almost Poisson brackets (see Section 3), in the context of non-holonomically constrained mechanical systems.

In this paper we go all the way in this direction and we work with a bracket, first introduced in [18], that is just required to be bilinear and a derivation on each of its entries. The derivation property, also known as the Leibniz rule, justifies why we refer to this structure as *Leibniz bracket*. The properties exhibited by the dynamical systems defined in this way are in general very different from those presented by standard Hamiltonian systems since most of the features of those systems are based on the Lie algebraic properties of the bracket that we have chosen to drop. This construction should not be mistaken with the Leibniz structures (also called Loday algebras) introduced by Loday [20] in the algebraic context.

The introduction of the notion of Leibniz system, whose elementary properties are presented in Section 2, is justified by the great variety of relevant systems whose natural underlying mathematical structure seems to be based in this kind of brackets. A number of these systems are described in Sections 3 and 4. To be more specific, in Section 3 we have identified the Leibniz structure inherent to a number of systems that can be found in the literature and Section 4 is devoted to the Leibniz formulation of simple mechanical systems subjected to non-holonomic constraints. One of the main differences between our treatment of this problem and other bracket formulations for non-holonomic systems is the fact that our bracket is defined in the entire phase space of the unconstrained system, unlike other approaches that provide a bracket only on the constraint submanifold. The Leibniz bracket that we construct in that section associates to the Hamiltonian of the unconstrained system a vector field that, when restricted to the constraint submanifold, coincides with the evolution vector field of the constrained system. As we point out in Section 4, the construction of this bracket involves the use of certain extensions that make it not to be uniquely determined by the dynamics that we want to describe. This leeway can be used in specific examples (see Section 4) to encode in a single bracket entire families of constraints, which may be very useful in the study of bifurcation problems in the non-holonomic context.

Section 5 contains a first approach to the study of the symmetries and the reduction of a Leibniz system. We introduce a notion of momentum map associated to symmetries that do not necessarily preserve the Leibniz bracket but that nevertheless produce non-trivial conservation laws as long as the Hamiltonian function is invariant. We also formulate a theorem that spells out, under certain regularity assumptions, how the orbit space of a symmetry that does respect the Leibniz structure (in a weak sense that is introduced in the text) is again a Leibniz manifold. This result generalizes to the context of the weak symmetries of a Leibniz manifold the well known result for the strong symmetries of a Poisson manifold. The analog of symplectic or Marsden–Weinstein reduction (see [25]) in this context is the subject of ongoing research and will be treated elsewhere.

Finally, Section 6 contains the generalization to the Leibniz context of the Poisson reduction results in [4,24,27] that characterize the situations in which a new Leibniz structure can be obtained by restriction to a subset and projection to the orbit space of a (pseudo)group of symmetries or to the leaf space of a distribution.

2. Leibniz systems

Definition 2.1. Let P be a smooth manifold and let $C^\infty(P)$ be the ring of smooth functions on it. A *Leibniz bracket* on P is a bilinear map $[\cdot,\cdot] : C^\infty(P) \times C^\infty(P) \to C^\infty(P)$ that is a derivation on each entry, that is:

$$[fg, h] = [f, h]g + f[g, h] \quad \text{and} \quad [f, gh] = g[f, h] + h[f, g]$$

for any $f, g, h \in C^\infty(P)$. We will say that the pair $(P, [\cdot,\cdot])$ is a *Leibniz manifold*. If the bracket $[\cdot,\cdot]$ is antisymmetric, that is, it satisfies

$$[f, g] = -[g, f]$$

for every pair of functions $f, g \in C^\infty(P)$ then we say that $(P, [\cdot,\cdot])$ is an *almost Poisson manifold*. We will usually denote the almost Poisson brackets with the symbol $\{\cdot,\cdot\}$.

A function $f \in C^\infty(P)$ such that $[f, g] = 0$ (respectively, $[g, f] = 0$) for any $g \in C^\infty(P)$ is called a *left* (respectively, *right*) *Casimir* of the Leibniz manifold $(P, [\cdot,\cdot])$.

Definition 2.2. Let $(P, \{\cdot,\cdot\})$ be an almost Poisson manifold. We define the *Jacobiator* of the bracket $\{\cdot,\cdot\}$ as the map $\mathfrak{J} : C^\infty(P) \times C^\infty(P) \times C^\infty(P) \to C^\infty(P)$ given by

$$\mathfrak{J}(f, g, h) = \{\{f, g\}, h\} + \{\{g, h\}, f\} + \{\{h, f\}, g\}. \tag{2.1}$$

A *Poisson structure* on P is an almost Poisson structure on P for which the Jacobiator is the zero map.

The following remarks are a direct consequence of the fact that the Leibniz structure is a derivation.

Let $(P, [\cdot,\cdot])$ be a Leibniz manifold and let h be a smooth function on P. There exist two vector fields X_h^{R} and X_h^{L} on P uniquely characterized by the relations:

$$X_h^{\mathrm{R}}[f] = [f, h] \quad \text{and} \quad X_h^{\mathrm{L}}[f] = -[h, f] \quad \text{for any } f \in C^\infty(P).$$

Given two smooth functions $g, h \in C^\infty(P)$ there exists a unique vector field $X_{g,h}$ on P such that

$$X_{g,h}[f](m) = \mathfrak{J}(f, g, h)(m) \quad \text{for any } f \in C^\infty(P).$$

We will call X_h^{R} the *Leibniz vector field* associated to the *Hamiltonian function* $h \in C^\infty(P)$. In this paper, the abbreviation X_h will always denote X_h^{R}. The flow F_t of the vector field X_h satisfies

$$\left.\frac{\mathrm{d}}{\mathrm{d}t}\right|_{t=0} g(F_t(m)) = [g, h](F_t(m)) \quad \text{for any } g \in C^\infty(P). \tag{2.2}$$

A straightforward corollary of (2.2) is that in the context of almost Poisson manifolds the Hamiltonian function is a constant of motion, that is, if F_t is flow of X_h then $h \circ F_t = h$

for any $h \in C^\infty(P)$. Note that since $[\cdot,\cdot]$ and $\mathfrak{J}$ are a derivation on each of their arguments they only depend on the first derivatives of the functions and thus, we can define two tensor maps $B : T^*P \times T^*P \to \mathbb{R}$ and $B_J : T^*P \times T^*P \times T^*P \to \mathbb{R}$ by

$$B(\mathbf{d}f, \mathbf{d}g) = [f, g] \quad \text{and} \quad B_J(\mathbf{d}f, \mathbf{d}g, \mathbf{d}h) = \mathfrak{J}(f, g, h) \tag{2.3}$$

for any $f, g, h \in C^\infty(P)$. We will refer to $B : T^*P \times T^*P \to \mathbb{R}$ as the *Leibniz tensor* associated to Leibniz bracket $[\cdot,\cdot]$. Conversely, any $(0, 2)$-tensor $B \in \mathcal{T}^2_0(P)$ defines a Leibniz bracket $[\cdot,\cdot]$ on $C^\infty(P)$ via the first equality in (2.3). These two structures will be used interchangeably. We can associate to the tensor B two vector bundle maps $B^\sharp_{\mathrm{L}} : T^*P \to TP$ and $B^\sharp_{\mathrm{R}} : T^*P \to TP$ defined by the relations:

$$B(\alpha, \beta) = -\langle \beta, B^\sharp_{\mathrm{L}}(\alpha)\rangle \quad \text{and} \quad B(\alpha, \beta) = \langle \alpha, B^\sharp_{\mathrm{R}}(\beta)\rangle$$

for any $\alpha, \beta \in T^*P$. Notice that when the bracket $[\cdot,\cdot]$ is symmetric (respectively, antisymmetric) we have that $B^\sharp_{\mathrm{R}} = -B^\sharp_{\mathrm{L}}$ (respectively, $B^\sharp_{\mathrm{R}} = B^\sharp_{\mathrm{L}}$) and $X^{\mathrm{R}}_h = -X^{\mathrm{L}}_h$ (respectively, $X^{\mathrm{R}}_h = X^{\mathrm{L}}_h$) for any $h \in C^\infty(P)$. We say that the Leibniz manifold $(P, [\cdot,\cdot])$ is *non-degenerate* whenever the maps $B^\sharp_{\mathrm{L}}$ and $B^\sharp_{\mathrm{R}}$ are vector bundle isomorphisms. All along this paper, the abbreviation $B^\sharp$ will always denote $B^\sharp_{\mathrm{R}}$.

Definition 2.3. Let $(P, [\cdot,\cdot])$ be a Leibniz manifold. We define the *left* and *right characteristic distributions* $\mathcal{E}_{\mathrm{L}}$ and $\mathcal{E}_{\mathrm{R}}$, respectively, by

$$\mathcal{E}_{\mathrm{L}} := \mathrm{span}\{X^{\mathrm{L}}_h | h \in C^\infty(P)\} = B^\sharp_{\mathrm{L}}(T^*P) \quad \text{and}$$
$$\mathcal{E}_{\mathrm{R}} := \mathrm{span}\{X^{\mathrm{R}}_h | h \in C^\infty(P)\} = B^\sharp_{\mathrm{R}}(T^*P).$$

Notice that if the Leibniz bracket $[\cdot,\cdot]$ is either symmetric or antisymmetric then both distributions coincide. If additionally the Leibniz manifold $(P, [\cdot,\cdot])$ is non-degenerate then $\mathcal{E}_{\mathrm{L}} = \mathcal{E}_{\mathrm{R}} = TP$ and we can define a tensor field $\omega : TP \times TP \to \mathbb{R}$ of type $(0, 2)$ on P by

$$\omega(X_f, X_g) = [f, g] \tag{2.4}$$

for any $f, g \in C^\infty(P)$. Given any point $m \in P$ and any vector subspace $V \subset T_mP$ we denote

$$V^\omega := \{w \in TP | \omega(m)(v, w) = 0 \ \text{ for any } \ v \in V\}.$$

If the tensor ω is antisymmetric (respectively, symmetric) then it is a two-form (respectively, a pseudo-Riemannian metric) on P. If additionally the form ω is closed we say that ω is a *symplectic form* on P and that the pair (P, ω) is a *symplectic manifold*.

Two smooth functions $h_1, h_2 \in C^\infty(P)$ on the Leibniz manifold $(P, [\cdot,\cdot])$ are said to be *equivalent* if and only if $[f, h_1 - h_2] = 0$ for any $f \in C^\infty(P)$ or equivalently, whenever $X_{h_1} = X_{h_2}$. Notice that this definition establishes an equivalence relation on the set $C^\infty(P)$.

Definition 2.4. A *Leibniz map* between two Leibniz manifolds $(P_1, [\cdot,\cdot]_1)$ and $(P_2, [\cdot,\cdot]_2)$ is a smooth map $\phi : P_1 \to P_2$ that satisfies

$$\phi^*[f, g]_2 = [\phi^* f, \phi^* g]_1 \quad \text{for any} \ \ f, g \in C^\infty(P_2).$$

Lemma 2.5. *Let $\phi : (P_1, [\cdot,\cdot]_1) \to (P_2, [\cdot,\cdot]_2)$ be a Leibniz map. Let $h \in C^\infty(P_2)$, F_t^2 be the flow of the Leibniz vector field X_h, F_t^1 the flow of $X_{h\circ\phi}$, and $\mathrm{Dom}(F_t^1)$ and $\mathrm{Dom}(F_t^2)$ the domains of definition of F_t^1 and F_t^2, respectively. Then $\mathrm{Dom}(F_t^1) \subset \phi^{-1}(\mathrm{Dom}(F_t^2))$ and*

$$F_t^2 \circ \phi(z) = \phi \circ F_t^1(z) \quad \textit{for any } z \textit{ in the domain } \mathrm{Dom}(F_t^1) \textit{ of } F_t^1. \tag{2.5}$$

Additionally, $X_{h\circ\phi}$ and X_h are ϕ-related, that is, $T\phi \circ X_{h\circ\phi} = X_h \circ \phi$.

Proof. Let $z \in \mathrm{Dom}(F_t^1)$ and $g \in C^\infty(P_2)$ arbitrary. Using the Leibniz condition on the map ϕ we can write

$$\begin{aligned}
&\mathbf{d}g((\phi \circ F_t^1)(z)) \cdot \frac{\mathrm{d}}{\mathrm{d}t}(\phi \circ F_t^1)(z) \\
&\quad = \frac{\mathrm{d}}{\mathrm{d}t} g((\phi \circ F_t^1)(z)) = \frac{\mathrm{d}}{\mathrm{d}t}(g \circ \phi)(F_t^1(z)) = [g \circ \phi, h \circ \phi](F_t^1(z)) \\
&\quad = [g, h](\phi \circ F_t^1)(z) = \mathbf{d}g((\phi \circ F_t^1)(z)) \cdot X_h((\phi \circ F_t^1)(z)).
\end{aligned}$$

Since the function g is arbitrary, this equality implies that

$$\frac{\mathrm{d}}{\mathrm{d}t}(\phi \circ F_t^1)(z) = X_h((\phi \circ F_t^1)(z)),$$

which allows us to conclude that $(\phi \circ F_t^1)(z)$ is an integral curve of X_h through the point $\phi(z)$. Since F_t^2 is the flow of X_h this automatically implies that $\phi(z) \in \mathrm{Dom}(F_t^2)$. As $z \in \mathrm{Dom}(F_t^1)$ is arbitrary we get that $\phi(\mathrm{Dom}(F_t^1)) \subset \mathrm{Dom}(F_t^2)$ which implies that $\mathrm{Dom}(F_t^1) \subset \phi^{-1}(\mathrm{Dom}(F_t^2))$. Additionally, the uniqueness property of the flow of a smooth vector field allows us to write that $(\phi \circ F_t^1)(z) = (F_t^2 \circ \phi)(z)$.

The ϕ-relatedness of $X_{h\circ\phi}$ and X_h follows from taking the time derivative of (2.5) at $t = 0$, recalling that $\mathrm{Dom}(F_t^1)$ becomes the entire manifold P_1 when t goes to zero. □

Proposition 2.6. *Let $\phi : (P, \{\cdot,\cdot\}_P) \mapsto (Q, [\cdot,\cdot]_Q)$ be a surjective Leibniz map. If $(P, \{\cdot,\cdot\}_P)$ is a Poisson manifold then so is $(Q, [\cdot,\cdot]_Q)$.*

Proof. Let $f, g \in C^\infty(Q)$ arbitrary. The surjectivity of ϕ implies that any element in Q can be written as $\phi(z)$, for some $z \in P$. Hence,

$$[f, g]_Q(\phi(z)) = \{f \circ \phi, g \circ \phi\}_P(z) = -\{g \circ \phi, f \circ \phi\}_P(z) = -[g, f]_Q(\phi(z)),$$

which proves the antisymmetry of $[\cdot,\cdot]_Q$. Analogously, in order to prove that $[\cdot,\cdot]_Q$ satisfies the Jacobi identity consider $f, g, h \in C^\infty(Q)$ and $z \in P$. Since ϕ is a Leibniz map we can write

$$\begin{aligned}
&[f, [g, h]_Q]_Q(\phi(z)) + [g, [h, f]_Q]_Q(\phi(z)) + [h, [f, g]_Q]_Q(\phi(z)) \\
&\quad = \{f \circ \phi, \{g \circ \phi, h \circ \phi\}_P\}_P + \{g \circ \phi, \{h \circ \phi, f \circ \phi\}_P\}_P \\
&\qquad + \{h \circ \phi, \{f \circ \phi, g \circ \phi\}_P\}_P(z) = 0.
\end{aligned}$$

□

3. Examples

As we already said symplectic and Poisson manifolds are particular cases of Leibniz manifolds. We now briefly introduce other non-trivial examples.

(i) Pseudometric brackets and gradient dynamical systems. Let $g : TP \times TP \to \mathbb{R}$ be a pseudo-Riemannian metric on the smooth manifold P, that is, a symmetric non-degenerate tensor field of type $(0, 2)$ on P. Let $g^\sharp : T^*P \to TP$ and $g^\flat : TP \to T^*P$ be the associated vector bundle maps. Given any smooth function $h \in C^\infty(P)$ we define its gradient $\nabla h : P \to TP$ as the vector field on P given by $\nabla h := g^\sharp\, \mathbf{d}h$. Let $[\cdot, \cdot] : C^\infty(P) \times C^\infty(P) \to \mathbb{R}$ be the Leibniz bracket defined by

$$[f, h] := g(\nabla f, \nabla h)$$

for any $f, h \in C^\infty(P)$. We will refer to this bracket as the *pseudometric bracket* associated to g. This bracket is clearly symmetric and non-degenerate and the Leibniz vector field X_h associated to any function $h \in C^\infty(P)$ is such that $X_h = \nabla h$. These brackets are also called *Beltrami brackets*, see [16,32]. Gradient systems appear profusely in the context of control, dynamical systems, and circuit theory (see [6,11,15,16,29], and references therein).

(ii) The three-wave interaction. A very relevant problem in dynamics is the study of the interaction between non-linear oscillators and the energy exchange between them. This problem can be viewed as an interaction between waves of different frequencies with different resonance conditions. A particular case that has deserved special attention is the so called three–wave or triad interaction [3]. Following [7] this problem can be formulated as a dynamical system in $\mathbb{R}^3$ that satisfies the differential equations given by

$$\frac{\mathrm{d}x}{\mathrm{d}t} = s_1\gamma_1 yz, \qquad \frac{\mathrm{d}y}{\mathrm{d}t} = s_2\gamma_2 xz, \qquad \frac{\mathrm{d}z}{\mathrm{d}t} = s_3\gamma_3 xy,$$

where the parameters $s_1, s_2, s_3 \in \{-1, 1\}$ and $\gamma_1, \gamma_2, \gamma_3$ are real numbers that satisfy $\gamma_1 + \gamma_2 + \gamma_3 = 0$. This system happens to be a particular case of point (i) by taking the Leibniz bracket induced by the constant pseudo-Riemannian metric:

$$g = \begin{pmatrix} \dfrac{1}{s_1\gamma_1} & 0 & 0 \\ 0 & \dfrac{-1}{s_2\gamma_2} & 0 \\ 0 & 0 & \dfrac{1}{s_3\gamma_3} \end{pmatrix}$$

and the Hamiltonian function $H(x, y, z) = xyz$.

(iii) Double bracket dissipation. As we already said the Leibniz dynamical systems induced by an almost Poisson bracket are energy preserving. Morrison [26] and Brockett [12,13] have proposed the modeling of certain dissipative phenomena by adding a symmetric bracket to a known antisymmetric one, that is:

$$[\cdot, \cdot]_{\mathrm{Leibniz}} = \{\cdot, \cdot\}_{\mathrm{skew}} + [\cdot, \cdot]_{\mathrm{sym}},$$

where the bracket $\{\cdot, \cdot\}_{\mathrm{skew}}$ is skewsymmetric, $[\cdot, \cdot]_{\mathrm{sym}}$ is symmetric, and hence the sum is a Leibniz bracket. This scheme allows the modeling of a surprising number of physical examples. The reader is encouraged to check with [10,23] for an account of applications and references in this direction.

A particularly simple example that fits into this framework is the equation arising from the Landau–Lifschitz model for the magnetization vector $\mathbf{M}$ in an external vector field $\mathbf{B}$:

$$\dot{\mathbf{M}} = \gamma \mathbf{M} \times \mathbf{B} + \frac{\lambda}{\|\mathbf{M}\|^2}(\mathbf{M} \times (\mathbf{M} \times \mathbf{B})), \tag{3.1}$$

where γ and λ are physical parameters. This equation is Leibniz in our sense if we take the Leibniz bracket on $\mathbb{R}^3$ given by the sum of the two brackets:

$$\{f, g\}_{\mathrm{skew}}(\mathbf{M}) := \mathbf{M} \cdot (\nabla f(\mathbf{M}) \times \nabla g(\mathbf{M})) \quad \text{and}$$
$$[f, g]_{\mathrm{sym}}(\mathbf{M}) := \frac{\lambda(\mathbf{M} \times \nabla f(\mathbf{M}))(\mathbf{M} \times \nabla g(\mathbf{M}))}{\gamma \|\mathbf{M}\|^2},$$

where the symbol $\times$ denotes the standard cross product on $\mathbb{R}^3$ and ∇ is the Euclidean gradient. With this bracket the differential equation (3.1) corresponds to the expression of the Leibniz vector field determined by the function:

$$h(\mathbf{M}) = \gamma \mathbf{B} \cdot \mathbf{M}.$$

Another related example is the differential equation satisfied by a rigid body subjected to certain dissipation (see [23]):

$$\dot{\mathbf{M}} = \mathbf{M} \times \boldsymbol{\Omega} + \alpha(\mathbf{M} \times (\mathbf{M} \times \boldsymbol{\Omega})). \tag{3.2}$$

In this expression $\mathbf{M}$ is the momentum vector of the solid and $\boldsymbol{\Omega}$ its angular velocity, both in body coordinates. Recall that

$$\boldsymbol{\Omega} := \left(\frac{M_1}{I_1}, \frac{M_2}{I_2}, \frac{M_3}{I_3}\right),$$

where (I_1, I_2, I_3) are the components of the inertia tensor of the body with respect to a basis in which this tensor is diagonal. If we take the same bracket as before with $\alpha = \lambda/\gamma\|\mathbf{M}\|^2$, Eq. (3.2) coincide with the Hamilton equations corresponding to the function:

$$h(\mathbf{M}) = \frac{1}{2}\left(\frac{M_1^2}{I_1} + \frac{M_2^2}{I_2} + \frac{M_3^2}{I_3}\right).$$

(iv) Almost Poisson manifolds and non-holonomically constrained mechanical systems. The equations of motion of a simple mechanical system subjected to a constraint can be written using D'Alembert's Principle. When the constraints can be expressed as a linear function on the velocities these equations admit a Leibniz formulation that corresponds to the almost Poisson bracket introduced in Definition 2.1. See [8,9,14,17,21,31,33] and references therein. If the constraints do not satisfy the linearity condition the almost Poisson formulation ceases to be valid in general. Nevertheless, if the constraints are affine on the velocities the problem still admits a formulation in the context of Leibniz manifolds using a bracket that in general is not antisymmetric. We discuss this point in detail in the following section.

4. Example: non-holonomic constraints

Let us consider a simple mechanical system characterized by a hyperregular Lagrangian $L : TQ \to \mathbb{R}$ on the tangent bundle TQ of a configuration space Q. One way to impose kinematic constraints on that system consists of fixing an affine subbundle $C \subset TQ$, usually referred to as the set of admissible kinematical states. The hyperregularity of L implies that the associated Legendre transform $\mathbb{F}L : TQ \to T^*Q$ is a diffeomorphism that can be used to define an associated Hamiltonian dynamical system on T^*Q with Hamiltonian function H, as well as the *Hamiltonian constraint submanifold* $D := \mathbb{FL}(C)$ on T^*Q. Let

$$T_D(T^*Q) := \{v_\alpha \in T(T^*Q) | \alpha \in D\} = T(T^*Q)|_D$$

be the restricted bundle. D'Alembert's Principle defines (see [21,22]) a vector subbundle $W \subset T_D(T^*Q)$ such that if $TD \cap W = \{0\}$ and the Hamiltonian vector field $X_H|_D$ is a section of $TD \oplus W$ then the corresponding splitting

$$X_H|_D = X_D^H + X_W^H$$

is well defined and X_D^H is the vector field whose flow describes the motion of the constrained dynamical system. The vector field X_D^H is usually referred to as the *evolution vector field* and the complementary vector field X_W^H as the *constraint force field*.

Our goal in the following paragraphs consists of endowing T^*Q with a Leibniz structure $[\cdot,\cdot]$ such that the Leibniz vector field X_H^{R} associated to H is such that

$$X_H^{\mathrm{R}}(z) = X_D^H(z) \quad \forall z \in D.$$

Theorem 4.1. *Assume T^*Q paracompact. Let $H \in C^\infty(T^*Q)$ be a smooth function with Hamiltonian vector field associated X_H. Let $D \subset T^*Q$ be a closed and embedded constraint submanifold and $W \subset T_D(T^*Q)$ a smooth vector subbundle such that $T_D(T^*Q) = TD \oplus W$. There exists a Leibniz structure $[\cdot,\cdot]$ on T^*Q such that*

$$X_H^{\mathrm{R}}(z) = \pi X_H(z) =: X_D^H(z), \quad z \in D, \tag{4.1}$$

where $\pi : TD \oplus W \to TD \oplus W$ is the natural projection onto the TD summand.

Proof. Consider the bilinear mapping:

$$[\cdot,\cdot]_D : C^\infty(T^*Q) \times C^\infty(T^*Q) \to C^\infty(D)$$

defined by $[f, g]_D(z) := \langle \mathbf{d}f(z), \pi B^\sharp(z)(\mathbf{d}g(z))\rangle$ for any $z \in D$, and where $B^\sharp : T^*(T^*Q) \to T^*(T^*Q)$ is the vector bundle isomorphism induced by the canonical symplectic form of T^*Q. This bracket has a smooth section $\tilde{B}_D : D \to \mathcal{T}^2_0(T^*Q)$ associated given by

$$\tilde{B}_D(z)(\alpha_z, \beta_z) := \langle \alpha_z, \pi B^\sharp(z)(\beta_z)\rangle$$

for any $\alpha_z, \beta_z \in T_z^*(T^*Q)$. By the smooth Tietze extension theorem (see for instance [2, Theorem 5.5.9]), $\tilde{B}_D$ can be extended to a smooth section $\tilde{B} : T^*Q \to \mathcal{T}^2_0(T^*Q)$. For any $f, g \in C^\infty(T^*Q)$ we define

$$[f, g]_{\mathrm{L}}(m) = \tilde{B}(m)(\mathbf{d}f(m), \mathbf{d}g(m)).$$

The bracket $[\cdot,\cdot]_{\mathrm{L}} : C^\infty(T^*Q) \times C^\infty(T^*Q) \to C^\infty(T^*Q)$ endows T^*Q with a Leibniz structure. Finally, we show that (4.1) holds. Indeed, for any $z \in D$:

$$X_H^{\mathrm{R}}(z) = \tilde{B}^\sharp(z)(\mathbf{d}H(z)) = \tilde{B}_D(z)(\mathbf{d}H(z)) = \pi B^\sharp(z)(\mathbf{d}H(z)) = X_D^H(z). \qquad \square$$

Remark 4.2. The Leibniz structures on T^*Q for which (4.1) holds are not unique. This freedom in the construction of the bracket can be used in specific applications to study families of systems instead of just a particular one, which may be of relevance in bifurcation theoretical problems. We make this comment more specific with the following elementary example.

Consider a particle of mass m constrained to move in a rotating hoop of mass M whose axis of rotation is parallel to the gravity and contains a diameter of the hoop. Let ϕ be the angle that parameterizes the position of the bead in the hoop and ψ the angle that characterizes the position of the hoop. This setup can be seen as a simple mechanical system with configuration space the two torus $\mathbb{T}^2$ and subjected to the affine constraint $\dot{\psi} - \omega = 0$, where $\omega \in \mathbb{R}$ is the constant angular speed of the hoop.

From the point of view of the formalism that we introduced above, for each value ω, there is a Hamiltonian constraint submanifold, D_ω and a vector subbundle $W = \operatorname{span}\{\partial/\partial p_\psi\}$ provided by D'Alembert's principle (see [21]) such that

$$T(T^*\mathbb{T}^2) = TD_\omega \oplus W.$$

As in the proof of the theorem, there exists for each ω a smooth section $\tilde{B}_{D_\omega} : D_\omega \to \mathcal{T}^2_0(T^*\mathbb{T}^2)$. Given that the spaces D_ω, $\omega \in \mathbb{R}$ form a foliation of $T^*\mathbb{T}^2$ we can define the section $\tilde{B} : T^*\mathbb{T}^2 \to \mathcal{T}^2_0(T^*\mathbb{T}^2)$ by $\tilde{B}(z) = \tilde{B}_{D_\omega}(z)$, where $z \in D_\omega$. The bracket $[\cdot,\cdot]_{\mathrm{L}}$ induced by $\tilde{B}$ on $T^*\mathbb{T}^2$ can be used to describe the system for *any* value of the angular velocity of the hoop. On other words if $H \in C^\infty(\mathbb{T}^2)$ is the Hamiltonian of the system then $X_H^{\mathrm{R}}(z) = X^H_{D_\omega}(z)$ for any $z \in D_\omega$ and any $\omega \in \mathbb{R}$.

Remark 4.3. There are other brackets that can be found in the literature in the context of mechanical systems with non-holonomic constraints. Most of them are equivalent to a construction given in [33] that roughly consists of modifying the proof of Theorem 4.1 by considering the bilinear mapping $[\cdot,\cdot]_D : C^\infty(T^*Q) \times C^\infty(T^*Q) \to C^\infty(D)$ defined by

$$[f, g]_D(z) := \langle \mathbf{d}f(z), \pi \circ B^\sharp(z) \circ \pi^*(\mathbf{d}g(z))\rangle = B(z)(\pi^*(\mathbf{d}f(z)), \pi^*(\mathbf{d}g(z)))$$

for any $z \in D$ and where π^* is the dual of the vector bundle map π. This skew symmetric bilinear map can be extended to an almost Poisson bracket $\{\cdot,\cdot\}$ on T^*Q, using, as we did in Theorem 4.1, the smooth Tietze extension theorem. The bracket $\{\cdot,\cdot\}$ is in general different from the Leibniz structure $[\cdot,\cdot]$ introduced in Theorem 4.1.

When the constraint subbundle C is *linear*, the vector fields X_H and X_H^{R} associated to the Hamiltonian of the system using the brackets $\{\cdot,\cdot\}$ and $[\cdot,\cdot]$, respectively, coincide in D and the corresponding flow describes the actual dynamics of the constrained system. Indeed, when the non-holonomic constraint is linear on the velocities, we can use the fundamental fact that $\mathbf{d}H|_D \in W^\circ$ (see [22]) or, equivalently, that $\pi^*\,\mathbf{d}H(z) = \mathbf{d}H(z)$ for any $z \in D$. Consequently,

$$X_H(z) = \pi \circ B^\sharp(z) \circ \pi^*(\mathbf{d}H(z)) = \pi \circ B^\sharp(z)(\mathbf{d}H(z)) = X_H^{\mathrm{R}}(z) = X_D^H(z).$$

When the constraint subbundle C is strictly *affine* the almost Poisson structure in [33] does *not* generate the vector field of the constrained system out of the Hamiltonian function. This has been pointed out in [14]. However, this is still true for the Leibniz bracket that we introduced in Theorem 4.1.

More detailed information on the relations among the different brackets in the literature can be found in [14,30], and references therein.

5. Symmetries and reduction of Leibniz systems

The symmetries of a dynamical system are in general very useful to simplify its study. In the particular case of symplectic and Poisson manifolds this idea has been specifically implemented using a procedure that is generically known as *reduction* (see [24,25,27,28] and references therein). In the next two sections, we will adapt some aspects of the reduction theory of symplectic and Poisson systems to the context of Leibniz manifolds.

In this section, we will consider symmetries of Leibniz systems that are encoded under the form of Lie group and Lie algebra actions. Let $\mathfrak{g}$ be a Lie algebra and P a smooth manifold. We recall that a *right (left) Lie algebra action* of $\mathfrak{g}$ on P is a Lie algebra (anti)homomorphism $\xi \in \mathfrak{g} \mapsto \xi_P \in \mathfrak{X}(P)$ such that the mapping $(m, \xi) \in P \times \mathfrak{g} \mapsto \xi_P(m) \in TP$ is smooth. The symbol $\mathfrak{X}(P)$ denotes the set of smooth vector fields on P. We will denote by $C^\infty(P)^\mathfrak{g}$ the set of $\mathfrak{g}$-invariant smooth functions on P, that is, $C^\infty(P)^\mathfrak{g} := \{f \in C^\infty(P) \,|\, \mathbf{d}f \cdot \xi_P = 0 \text{ for any } \xi \in \mathfrak{g}\}$.

Definition 5.1. Let $(P, [\cdot, \cdot])$ be a Leibniz manifold and $B \in \mathcal{T}^2_0(P)$ the associated Leibniz tensor. Let G be a Lie group (respectively, $\mathfrak{g}$ a Lie algebra) acting on P. We say that this G-action (respectively, $\mathfrak{g}$-action) is a *weak symmetry* of $(P, [\cdot, \cdot])$ whenever the algebra $C^\infty(P)^G$ (respectively, $C^\infty(P)^\mathfrak{g}$) of G-invariant functions on P is closed under the Leibniz bracket. We say that G (respectively, $\mathfrak{g}$) is a *strong symmetry* if G acts on P by Leibniz maps (respectively, if $\pounds_{\xi_P} B = 0$ for any $\xi \in \mathfrak{g}$). Such actions will be sometimes referred to as *canonical*.

Definition 5.2. Let $(P, [\cdot, \cdot])$ be a Leibniz manifold and $\mathfrak{g}$ a Lie algebra acting on P. We say that the $\mathfrak{g}$-action on P admits a *momentum map* $\mathbf{J} : P \to \mathfrak{g}^*$ whenever for any $\xi \in \mathfrak{g}$, there exists a smooth function $f_\xi \in C^\infty(P)$ such that the component $\mathbf{J}^\xi := \langle \mathbf{J}, \xi \rangle$ is also smooth and

$$X^{\mathrm{L}}_{\mathbf{J}^\xi} = f_\xi \xi_P.$$

We will call the function f_ξ the ξ-*integrating factor*.

Remark 5.3. The notion of momentum map that we just defined is, in principle, not related to the one introduced in [9]. That definition is based on a generalization to the non-holonomic context of Smale's formula for the momentum map associated to a lifted Lie group action on the tangent bundle of the phase space. Definition 5.2 is a generalization of the classical definition in the symplectic and Poisson context where the hypothesis on the canonical character of the action has been dropped. As we see in the next proposition, even without this hypothesis, the level sets of the momentum map are conserved by the dynamics generated by any invariant Hamiltonian (*Noether's theorem*). We recall that this is not the case for the momentum map introduced in [9] where the momentum evolution is governed by the so called *momentum equation*.

Proposition 5.4. *Let* $(P, [\cdot,\cdot])$ *be a Leibniz manifold and* $\mathfrak{g}$ *a Lie algebra acting on P. Assume that the* $\mathfrak{g}$*-action on P admits a momentum map* $\mathbf{J} : P \to \mathfrak{g}^*$. *Then the level sets of the momentum map are preserved by the flows of the Leibniz vector fields associated to any* $\mathfrak{g}$*-invariant function on P.*

Proof. Let $h \in C^\infty(P)^{\mathfrak{g}}$ and $\xi \in \mathfrak{g}$ be arbitrary. Let F_t be the flow of the Leibniz vector field X_h^{R}. Then

$$\frac{\mathrm{d}}{\mathrm{d}t}\mathbf{J}^\xi \circ F_t = X_h^{\mathrm{R}}[\mathbf{J}^\xi] = [\mathbf{J}^\xi, h] = X^{\mathrm{L}}_{\mathbf{J}^\xi}[h] = f_\xi \xi_P[h] = f_\xi\, \mathbf{d}h \cdot \xi_P = 0. \qquad \square$$

Remark 5.5. The introduction of the integrating factors in the definition of the momentum map is not motivated by particular needs of the Leibniz category. Indeed, as we show in the following example, even in the symplectic or in the Poisson category this seems to be the only way to associate non-trivial conservation laws to non-canonical symmetries.

Let $\mathbb{R}^2_{\mathrm{u}}$ be the upper half plane considered as a symplectic manifold with form $\omega = \mathbf{d}x \wedge \mathbf{d}y$. Let $(\mathbb{R}, +)$ act on $\mathbb{R}^2_{\mathrm{u}}$ by $a \cdot (x, y) := (x, \mathrm{e}^a y)$ for any $(x, y) \in \mathbb{R}^2_{\mathrm{u}}$ and any $a \in \mathbb{R}$. This action is clearly not canonical but it still admits a momentum map with a non-trivial integrating factor that leads to a conservation law via Proposition 5.4. Indeed, for any $\xi \in \mathbb{R}$, the infinitesimal generator $\xi_{\mathbb{R}^2_{\mathrm{u}}}$ is given by $\xi_{\mathbb{R}^2_{\mathrm{u}}}(x, y) = (0, \mathrm{e}^\xi y)$, $(x, y) \in \mathbb{R}^2_{\mathrm{u}}$. This vector field is not Hamiltonian and hence this action does not have a traditional momentum map associated. However, there is a momentum map available in the sense of Definition 5.2 since the maps $\mathbf{J} : \mathbb{R}^2_{\mathrm{u}} \to \mathbb{R}$ and $f_\xi \in C^\infty(\mathbb{R}^2_{\mathrm{u}})$ given by $\mathbf{J}(x, y) := x$ and $f_\xi = -(\xi/\mathrm{e}^\xi y)$, $(x, y) \in \mathbb{R}^2_{\mathrm{u}}$, $\xi \in \mathbb{R}$, are such that $X_{\mathbf{J}^\xi} = f_\xi \xi_{\mathbb{R}^2_{\mathrm{u}}}$. The conservation law associated by Proposition 5.4 to the existence of this momentum map can be phrased by saying that the Hamiltonian flows corresponding to Hamiltonian functions that depend only on the x variable preserve the vertical lines in $\mathbb{R}^2_{\mathrm{u}}$.

The following result shows that the orbit space corresponding to the weak symmetry G of a Leibniz manifold $(P, [\cdot,\cdot])$ is also a Leibniz manifold provided that the group action satisfies enough regularity assumptions to guarantee that the quotient P/G is a regular quotient manifold, that is, P/G can be endowed with a (unique) smooth structure that makes the projection $\pi : P \to P/G$ a surjective submersion. This observation is consistent with the result of Bates and Sniatycki [5] in the context of non-holonomic mechanics that shows that symmetry reduction for these systems preserves the form of the equations of motion.

Theorem 5.6. *Let* $(P, [\cdot,\cdot])$ *be a Leibniz manifold and let G be a Lie group acting on P in such way that the orbit space* P/G *is a regular quotient manifold* (*this is the case when, for instance, the action is free and proper*). *Assume that G is a weak symmetry of* $(P, [\cdot,\cdot])$. *Then*

(i) $(P/G, [\cdot,\cdot]_{P/G})$ *is a Leibniz manifold with bracket* $[\cdot,\cdot]_{P/G}$ *uniquely determined by the expression*:

$$[f, g]_{P/G} \circ \pi = [\pi^* f, \pi^* g] \tag{5.1}$$

for any $f, g \in C^\infty(P/G)$ *and where* $\pi : P \to P/G$ *is the projection.*

(ii) *The Leibniz structure induced by the bracket* $[\cdot,\cdot]_{P/G}$ *on* P/G *is the only one for which the projection* $\pi : P \to P/G$ *is a Leibniz map.*

(iii) *Let* $h \in C^\infty(P)^G$ *be a smooth G-invariant function on P and* $h^{P/G} \in C^\infty(P/G)$ *the function on the quotient uniquely determined by the expression* $h^{P/G} \circ \pi = h$. *Let* X_h *and* $X_{h^{P/G}}$ *be the corresponding Leibniz vector fields on* $(P, [\cdot,\cdot])$ *and* $(P/G, [\cdot,\cdot]_{P/G})$, *respectively, and* F_t *and* $F_t^{P/G}$ *the associated flows. Then* $\mathrm{Dom}(F_t) \subset \pi^{-1}(\mathrm{Dom}(F_t^{P/G}))$ *and*

$$F_t^{P/G} \circ \pi(z) = \pi \circ F_t(z) \tag{5.2}$$

for any $z \in \mathrm{Dom}(F_t)$. *The vector fields* X_h *and* $X_{h^{P/G}}$ *are* π*-related.*

Proof. (i) We first check that (5.1) is a good definition for the bracket $[\cdot,\cdot]_{P/G}$. Let m, m' be two points in P such that $\pi(m) = \pi(m')$. This equality implies that there exists an element $g \in G$ such that $m' = g \cdot m$. Let now $f, g \in C^\infty(P)^G$ arbitrary. By definition $[f, g]_{P/G}(\pi(m')) = [f \circ \pi, g \circ \pi](m')$. Since by hypothesis $C^\infty(P)^G$ is closed under the bracket and $f \circ \pi$ and $g \circ \pi$ are G-invariant then so is $[f \circ \pi, g \circ \pi]$ and hence

$$\begin{aligned}[f, g]_{P/G}(\pi(m')) &= [f \circ \pi, g \circ \pi](m') = [f \circ \pi, g \circ \pi](g \cdot m) \\ &= [f \circ \pi, g \circ \pi](m) = [f, g]_{P/G}(\pi(m))\end{aligned}$$

as required. The bracket $[\cdot,\cdot]_{P/G}$ is clearly bilinear and is a derivation on its two arguments. Therefore, $(P/G, [\cdot,\cdot]_{P/G})$ is a Leibniz manifold. (ii) It is a consequence of the fact that the projection π is a surjective submersion. (iii) is a consequence of (ii) and Lemma 2.5. □

We emphasize that the weak symmetry condition on the Leibniz bracket $(P, [\cdot,\cdot])$ and the G-invariance of the Hamiltonian h do not suffice to ensure the G-equivariance of the associated Leibniz flow F_t of X_h. In general only (5.2) holds. The flow F_t is G-equivariant whenever the G-action is a strong symmetry of the bracket.

Remark 5.7. The second part of the theorem shows that if the G-action is a weak symmetry of a Leibniz manifold $(P, [\cdot,\cdot])$ then the quotient P/G admits a unique Leibniz structure $[\cdot,\cdot]_{P/G}$ with respect to which the projection $\pi : P \to P/G$ is a Leibniz map. The converse is also true. Indeed, let $f, g \in C^\infty(P)^G$ and $\bar{f}, \bar{g} \in C^\infty(P/G)$ be the unique smooth functions such that $\bar{f} \circ \pi = f$ and $\bar{g} \circ \pi = g$. Then the bracket $[f, g] \in C^\infty(P)$ is such that for any $h \in G$ and any $z \in P$:

$$\begin{aligned}[f, g](h \cdot z) &= [\bar{f} \circ \pi, \bar{g} \circ \pi](h \cdot z) = [\bar{f}, \bar{g}]_{P/G}(\pi(h \cdot z)) = [\bar{f}, \bar{g}]_{P/G}(\pi(z)) \\ &= [\bar{f} \circ \pi, \bar{g} \circ \pi](z) = [f, g](z),\end{aligned}$$

which proves that $[f, g] \in C^\infty(P)^G$ and hence that the G-action is a weak symmetry.

Remark 5.8. Proposition 2.6 guarantees that if the Leibniz manifold $(P, [\cdot,\cdot])$ in the previous theorem is actually Poisson then so is the reduced manifold $(P/G, [\cdot,\cdot]_{P/G})$.

Example 5.9. *(i) Double bracket dissipation.* Consider the systems with double bracket dissipation that we studied in Section 3. This time we will restrict our discussion to the points in $\mathbb{R}^3$ that do not lie in the third axis. Assume that the magnetic vector field $\mathbf{B}$ is constant and equal to the vector $(0,0,1)$ or, in the case of the rigid body subjected to a dissipative interaction, assume that the moments of inertia I_1 and I_2 are equal. In both cases, the rotations around the third axis, which is a free group action on the restricted phase space, leave invariant the Hamiltonian functions and constitute a strong symmetry for the Leibniz system which allows us to apply Theorem 5.6. For a concrete realization of the quotient Leibniz structure we use the invariant polynomials $\sigma_1 = (1/2)(M_1^2 + M_2^2)$ and $\sigma_2 = M_3$ of this action. In these coordinates the Leibniz tensor associated to the reduced Leibniz structure takes the form:

$$B = \frac{\gamma}{2\sigma_1 + \sigma_3^2}\begin{pmatrix} -2\sigma_1\sigma_2^2 & 2\sigma_1\sigma_2 \\ 2\sigma_1\sigma_2 & -2\sigma_1 \end{pmatrix}.$$

The reduced Hamiltonian functions are $h(\sigma_1,\sigma_2) = \gamma\sigma_2$ in the first case and $h = (\sigma_1/I_1) + (\sigma_2^2/2I_3)$ in the second.

(ii) Reduction of a Poisson system with a non-canonical symmetry. Consider the Poisson dynamical system $(\mathbb{R}^3_*, \{\cdot,\cdot\}, H)$ on $\mathbb{R}^3_* := \mathbb{R}^3 \setminus \{(x,y,0) \in \mathbb{R}^3\}$, where the Poisson bracket $\{\cdot,\cdot\}$ is determined by the Poisson tensor:

$$B(x,y,z) = \begin{pmatrix} 0 & x & y \\ -x & 0 & x \\ -y & -x & 0 \end{pmatrix}$$

for any $(x,y,z) \in \mathbb{R}^3_*$ and $H = (1/2)(x^2 + y^2)$. Let $G := (\mathbb{R}, +)$ act on $\mathbb{R}^3$ by the map $\phi : \mathbb{R} \times \mathbb{R}^3 \to \mathbb{R}^3$ given by

$$a \cdot (x,y,z) = (x, y, \mathrm{e}^a z) \quad \text{for any } a \in \mathbb{R} \text{ and any } (x,y,z) \in \mathbb{R}^3.$$

This action is not canonical since, for example, $\phi_a^*\{y,z\} = x \neq \mathrm{e}^a x = \{\phi_a^* y, \phi_a^* z\}$. However, notice that since the algebra of G-invariant functions is made by functions depending on just the first two variables, that is, $C^\infty(P)^G = \{f \in C^\infty(P) | f \equiv f(x,y)\}$ then $C^\infty(P)^G$ is closed under the Poisson bracket. Consequently, by Theorem 5.6, $(\mathbb{R}^3_*, \{\cdot,\cdot\}, H)$ can be reduced by this action. The reduced Poisson space is $\mathbb{R}^2$ with the Poisson structure given by the reduced Poisson tensor:

$$B_{\mathbb{R}^3_*/G}(x,y) = \begin{pmatrix} 0 & x \\ -x & 0 \end{pmatrix}$$

and the reduced Hamiltonian is $h(x,y) = (1/2)(x^2 + y^2)$.

(iii) A symmetry of the three-wave interaction. The Hamiltonian $H(x,y,z) = xyz$ of the three-wave interaction that we presented in Section 3 is invariant with respect to the action of

the product of the two multiplicative groups $G := (\mathbb{R}^+, \cdot) \times (\mathbb{R}^+, \cdot)$ by $(\lambda_1, \lambda_2) \cdot (x, y, z) := (\lambda_1 x, \lambda_2 y, \lambda_3 z)$, where $(x, y, z) \in \mathbb{R}^3$, $(\lambda_1, \lambda_2) \in \mathbb{R}^+ \times \mathbb{R}^+$, and $\lambda_3 := (\lambda_1 \cdot \lambda_2)^{-1}$. The infinitesimal generator for this action is given by $\xi_{\mathbb{R}^3}(x, y, z) = (ax, by, -(a+b)z)$ for any $\xi := (a, b) \in \mathbb{R}^2$. Even though this action is not even a weak Leibniz symmetry we can associate to it a momentum map $\mathbf{J} : \mathbb{R}^3 \to \mathbb{R}^2$ given by

$$\mathbf{J}(x, y, z) = \left(\frac{1}{2} \left(\frac{x^2}{s_1\gamma_1} - \frac{z^2}{s_3\gamma_3} \right), \frac{-1}{2} \left(\frac{y^2}{s_2\gamma_2} + \frac{z^2}{s_3\gamma_3} \right) \right).$$

By Proposition 5.4, the components of $\mathbf{J}$ are constants of the motion for the flow of the Leibniz vector field X_H.

(iv) Non-holonomically constrained particle. Consider a free particle in $\mathbb{R}^3$. We will encode this setup as a Hamiltonian dynamical system on the cotangent bundle $T^*\mathbb{R}^3$ endowed with its canonical symplectic structure. We will denote by $B \in \Lambda^2(T^*\mathbb{R}^3)$ the Poisson tensor associated to this symplectic form. The Hamiltonian function of this system is $H(x, y, z, p_x, p_y, p_z) = (1/2)(p_x^2 + p_y^2 + p_z^2)$. Suppose that the particle is forced to satisfy the affine constraint $\dot{x} + y\dot{z} - a = 0$, where $a \in \mathbb{R}$. In this particular case, the Hamiltonian constrained submanifold is given by

$$D_a = \{(x, y, z, p_x, p_y, p_z) \in T^*\mathbb{R}^3 | p_x + yp_z - a = 0\}. \tag{5.3}$$

Consequently,

$$T_{(x,y,z,p_x,p_y,p_z)} D_a = \mathrm{span} \left\{ \frac{\partial}{\partial x}, \frac{\partial}{\partial y} + p_z \frac{\partial}{\partial p_x}, \frac{\partial}{\partial z}, \frac{\partial}{\partial p_y}, -y \frac{\partial}{\partial p_x} + \frac{\partial}{\partial p_z} \right\}.$$

Using D'Alembert's principle (see [21]) we choose the subbundle $W_a \subset T_{D_a}(T^*\mathbb{R}^3)$ given by

$$W_a(x, y, z, p_x, p_y, p_z) := \mathrm{span} \left\{ \frac{\partial}{\partial p_x} + y \frac{\partial}{\partial p_z} \right\}$$

that satisfies the regularity condition $T_{D_a}(T^*\mathbb{R}^3) = TD_a \oplus W_a$. We now follow the scheme introduced in Section 4. A straightforward computation shows that the projection $\pi_a : TD_a \oplus W_a \to TD_a$ and the composition $\tilde{B}^\sharp_{D_a} := \pi_a \circ B^\sharp$ are given, using canonical coordinates, by the matrices:

$$\pi(m) = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & \dfrac{-p_z}{1+y^2} & 0 & \dfrac{y^2}{1+y^2} & 0 & \dfrac{-y}{1+y^2} \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & \dfrac{-yp_z}{1+y^2} & 0 & \dfrac{-y}{1+y^2} & 0 & \dfrac{1}{1+y^2} \end{pmatrix},$$

$$\tilde{B}^{\sharp}_{D_a}(m) = \begin{pmatrix} 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ \dfrac{-y^2}{1+y^2} & 0 & \dfrac{y}{1+y^2} & 0 & \dfrac{-p_z}{1+y^2} & 0 \\ 0 & -1 & 0 & 0 & 0 & 0 \\ \dfrac{y}{1+y^2} & 0 & \dfrac{-1}{1+y^2} & 0 & \dfrac{-yp_z}{1+y^2} & 0 \end{pmatrix}, \tag{5.4}$$

where $m = (x, y, z, p_x, p_y, p_z)$. Following the ideas introduced in the Remark 4.2 and noticing that the expression of $\tilde{B}^{\sharp}_{D_a}$ does not depend on the parameter a, we can trivially extend $\tilde{B}_{D_a}$ to a Leibniz tensor $\tilde{B} \in \mathcal{T}^2_0(T^*\mathbb{R}^3)$ whose restriction to any $D_a, a \in \mathbb{R}$, coincides with $\tilde{B}_{D_a}$. Using this extension and the expressions in (5.4) we can write the evolution vector field of the constrained system as

$$X^H_D = \tilde{B}^{\sharp}\, \mathbf{d}H,$$

which in canonical coordinates reads

$$\dot{x} = p_x, \qquad \dot{y} = p_y, \qquad \dot{z} = p_z, \qquad \dot{p}_x = \frac{-p_z p_y}{1+y^2},$$
$$\dot{p}_y = 0, \qquad \dot{p}_z = \frac{-yp_z p_y}{1+y^2}. \tag{5.5}$$

Note that the constraint does not need to be included in the set of equations since for a set of initial conditions satisfying the constraint, the dynamics will preserve it automatically. From the point of view of the Leibniz formulation of the problem this remark can be phrased by saying that the function defining the constraint is a left Casimir of the Leibniz system $(T^*\mathbb{R}^3, \tilde{B})$.

A momentum map. The Hamiltonian H is symmetric with respect to the lifted action of the translations on the configuration space $\mathbb{R}^3$. The infinitesimal generators associated to this action are given by $\xi_{T^*\mathbb{R}^3}(x, y, z, p_x, p_y, p_z) = (\xi, 0)$ for any $\xi \in \mathbb{R}^3$. The lifted translations along the OY-axis admit a momentum map $\mathbf{J} : T^*\mathbb{R}^3 \to \mathbb{R}$ with respect to the Leibniz structure $(T^*\mathbb{R}^3, \tilde{B})$, given by

$$\mathbf{J}(x, y, z, p_x, p_y, p_z) = p_y + \phi(p_x + yp_z),$$

where ϕ is an arbitrary smooth real valued function. By Proposition 5.4 $\mathbf{J}$ is preserved by the integral curves of the evolution vector field X^H_D.

Reduction. Consider now the group $G := (\mathbb{R}^2, +)$ acting on $T^*\mathbb{R}^3$ by lifting the translations in the coordinates x and z. This action is a weak symmetry of $(T^*\mathbb{R}^3, \tilde{B})$ and thus we can apply Theorem 5.6. Let $(T^*\mathbb{R}^3/G, [\cdot, \cdot]_{T^*\mathbb{R}^3/G}, h)$ be the reduced system. $T^*\mathbb{R}^3/G$ can be identified with $\mathbb{R}^4$ since the points in this orbit space correspond to the elements of the form (y, p_x, p_y, p_z). Using this identification the reduced Hamiltonian can be written as $h = (1/2)(p_x^2 + p_y^2 + p_z^2)$ and the reduced Leibniz tensor $\tilde{B}_1$ is given by

$$\tilde{B}_1^\sharp(y, p_x, p_y, p_z) = \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & \dfrac{-p_z}{1+y^2} & 0 \\ -1 & 0 & 0 & 0 \\ 0 & 0 & \dfrac{-yp_z}{1+y^2} & 0 \end{pmatrix}.$$

This Leibniz structure has two independent left Casimirs $C_1^{\mathrm{l}}(y, p_x, p_y, p_z) = p_y$, $C_2^{\mathrm{l}}(y, p_x, p_y, p_z) = p_x + yp_z$ and two independent right Casimirs $C_1^{\mathrm{r}}(y, p_x, p_y, p_z) = p_x$, $C_2^{\mathrm{r}}(y, p_x, p_y, p_z) = p_z$. Consequently the new Hamiltonian function $\bar{h} := h - (1/2)((C_1^{\mathrm{r}})^2 + (C_2^{\mathrm{r}})^2)$, that is, $\bar{h}(y, p_x, p_y, p_z) = (1/2)p_y^2$, has the same evolution vector field than that of h. This equivalent Hamiltonian admits the symmetry of translations in the coordinates p_x and p_z and hence we can further reduce the system onto a two-dimensional Leibniz one with tensor:

$$\tilde{B}_2^\sharp(y, p_y) = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$$

and Hamiltonian function $h_2 = (1/2)p_y^2$.

6. The reduction of a presheaf of Leibniz algebras

The reduction theorem that we presented in the previous section contains extremely strong regularity hypotheses that allowed us to have a smooth orbit space onto which the Leibniz bracket and the corresponding equivariant dynamics can be dropped. When these hypotheses are not present, the orbit space is not smooth anymore but nevertheless, the Leibniz algebra, or more specifically, the *presheaf of Leibniz algebras* associated to the bracket admits, under certain circumstances, a projection to the quotient. The algebraic approach to reduction that we introduce in the following paragraphs has its origins in the works [4,27] carried out in the context of symmetric Poisson manifolds.

We recall that a *sheaf* $\mathcal{F}$ of functions on a topological space P is a map that assigns to any open set U a set of real valued functions $\mathcal{F}(U)$ which is an algebra under multiplication. In the definition it is also required that for every inclusion $V \subset U$ of open sets there is a given homomorphism $\mathrm{res}_V^U : \mathcal{F}(U) \to \mathcal{F}(V)$ called the *restriction* from U to V that satisfies the following conditions:

(SH1) $\mathcal{F}(\emptyset) = \{0\}$ and $\mathrm{res}_U^U : \mathcal{F}(U) \to \mathcal{F}(U)$ is the identity map.
(SH2) If $W \subset V \subset U$ are open sets, then $\mathrm{res}_W^V \circ \mathrm{res}_V^U = \mathrm{res}_W^U$.
(SH3) Let U be an open set and $\{V_i\}_{i\in I}$ an open covering of U. If $f \in \mathcal{F}(U)$ is such that the restriction $\mathrm{res}_{V_i}^U(f)$ of f to each V_i is 0, then $f = 0$.
(SH4) Let U be an open set, $\{V_i\}_{i\in I}$ an open covering of U, and let $f_i \in \mathcal{F}(V_i)$ be given for each $i \in I$. Suppose that the restrictions of f_i and f_j to $V_i \cap V_j$ are equal for all $i, j \in I$. Then there exists a unique $f \in \mathcal{F}(U)$ whose restriction to each V_i is f_i for all $i \in I$.

When the map $\mathcal{F}$ satisfies only properties (SH1) and (SH2) we say that $\mathcal{F}$ is a *presheaf*. The elements in $\mathcal{F}(U)$ are called the *sections* of $\mathcal{F}$ over U. The elements in $\mathcal{F}(P)$ are called *global sections*.

Definition 6.1. Let M be a topological space with a presheaf $\mathcal{F}$ of smooth functions. A *presheaf of Leibniz algebras* on $(P, \mathcal{F})$ is a map $[\cdot,\cdot]$ that assigns to each open set $U \subset M$ a bilinear operation $[\cdot,\cdot]_U : \mathcal{F}(U) \times \mathcal{F}(U) \to \mathcal{F}(U)$ such that the pair $(\mathcal{F}(U), [\cdot,\cdot]_U)$ is a Leibniz algebra. A presheaf of Leibniz algebras will be usually denoted as a triple $(P, \mathcal{F}, [\cdot,\cdot])$.

We say that the presheaf of Leibniz algebras $(P, \mathcal{F}, [\cdot,\cdot])$ is *non-degenerate* when if $f \in \mathcal{F}(U)$, is such that $[f, g]_{U\cap V} = 0$ for any $g \in \mathcal{F}(V)$ and any open set of V, then f is constant in the connected components of U.

Example 6.2. Any Leibniz manifold $(P, [\cdot,\cdot])$ has a natural sheaf of Leibniz algebras on its sheaf of smooth functions that associates to any open subset U of P the restriction $[\cdot,\cdot]|_U$ of $[\cdot,\cdot]$ to $C^\infty(U) \times C^\infty(U)$.

6.1. Leibniz reduction by pseudogroups

The main goal of this section is the presentation of a result that fully characterizes the situations in which the sheaf of Leibniz algebras in Example 6.2 behaves properly under restriction to subsets and projection to the orbit spaces of pseudogroups of local Leibniz diffeomorphisms of $(P, [\cdot,\cdot])$.

We start by introducing our terminology. Let P be a smooth manifold and $\mathrm{Diff}_L(P)$ the pseudogroup of local diffeomorphisms of P. More explicitly, the elements of $\mathrm{Diff}_L(P)$ are diffeomorphisms $F : \mathrm{Dom}(F) \subset P \to F(\mathrm{Dom}(F))$ of an open subset $\mathrm{Dom}(F) \subset P$ onto its image $F(\mathrm{Dom}(F)) \subset P$. We will denote the elements of $\mathrm{Diff}_L(P)$ as pairs $(F, \mathrm{Dom}(F))$. The local diffeomorphisms can be composed using the binary operation defined as

$$(G, \mathrm{Dom}(G)) \cdot (F, \mathrm{Dom}(F)) := (G \circ F, F^{-1}(\mathrm{Dom}(G)) \cap \mathrm{Dom}(F)) \tag{6.1}$$

for all $(G, \mathrm{Dom}(G)), (F, \mathrm{Dom}(F)) \in \mathrm{Diff}_L(P)$. It is easy to see that this operation is associative and has $(\mathbb{I}, P)$, the identity map of P, as (unique) two-sided identity element, which makes $\mathrm{Diff}_L(P)$ into a monoid (set with an associative operation which contains a two-sided identity element). Notice that only the elements in $\mathrm{Diff}(P) \subset \mathrm{Diff}_L(P)$ have an inverse since, in general, for any $(F, \mathrm{Dom}(F)) \in \mathrm{Diff}_L(P)$, we have that

$$(F^{-1}, F(\mathrm{Dom}(F))) \cdot (F, \mathrm{Dom}(F)) = (\mathbb{I}|_{\mathrm{Dom}(F)}, \mathrm{Dom}(F)), \tag{6.2}$$

$$(F, \mathrm{Dom}(F)) \cdot (F^{-1}, F(\mathrm{Dom}(F))) = (\mathbb{I}|_{F(\mathrm{Dom}(F))}, F(\mathrm{Dom}(F))). \tag{6.3}$$

Consequently, the only way to obtain the identity element $(\mathbb{I}, P)$ out of the composition of F with its inverse is having $\mathrm{Dom}(F) = P$. It follows from this argument that $\mathrm{Diff}(P)$ is the biggest subgroup contained in the monoid $\mathrm{Diff}_L(P)$ with respect to the composition law (6.1). In the sequel we will frequently encounter submonoids A of $\mathrm{Diff}_L(P)$ that satisfy the following property:

(PS) for any $F : \mathrm{Dom}(F) \to F(\mathrm{Dom}(F))$ in A there exists another element $F^{-1} : F(\mathrm{Dom}(F)) \to \mathrm{Dom}(F)$ also in A that satisfies the identities (6.2) and (6.3).

Such submonoids will be referred to as *pseudogroups* of $\mathrm{Diff}_{\mathrm{L}}(P)$. Recall that A being a submonoid implies that it is closed under composition and $(\mathbb{I}, P) \in A$. One of the important features of pseudogroups is that they have an associated orbit space. Indeed, if A is a pseudogroup we define the *orbit* $A \cdot m$ under A of any element $m \in P$ as the set $A \cdot m := \{F(m) | F \in A, \text{ such that } m \in \mathrm{Dom}(F)\}$. A being a pseudogroup implies that the relation *being in the same A-orbit* is an equivalence relation and induces a partition of P into A-orbits. The *space of A-orbits* will be denoted by P/A. If we endow the space of orbits P/A with the quotient topology, the projection $\pi_A : P \to P/A$ is a continuous and open map.

Let $S \subset P$ be a subset of P endowed with a topology $\mathcal{T}$ that in general does not coincide with the relative or subspace topology. The sheaf C^∞_P of smooth functions on P induces a quotient sheaf $C^\infty_{P/A}$ on the orbit space P/A. Consider now the subset

$$A_S := \{a \in A | a(s) \in S \text{ for any } s \in S \cap \mathrm{Dom}(a)\}.$$

All along this section we will assume that A_S is a subpseudogroup of A. This hypothesis will allow us to construct the quotients S/A_S and P/A_S. Given that the quotient S/A_S can be seen as a subset of P/A_S, there is a well defined presheaf of Whitney smooth functions W^∞_{S/A_S} on S/A_S induced by C^∞_{P/A_S}. We recall (see [28]) that for any open set $V \subset S/A_S$, the elements $f \in W^\infty_{S/A_S}(V)$ are characterized by the fact that if $\pi_S : S \to S/A_S$ is the projection onto orbit space then for any $m \in \pi_S^{-1}(V)$ there exists an open A_S-invariant neighborhood of m in P and $F \in C^\infty_P(U_m)^{A_S}$ such that

$$f \circ \pi_S|_{\pi_S^{-1}(V)\cap U_m} = F|_{\pi_S^{-1}(V)\cap U_m}. \tag{6.4}$$

We will say that F is a *local extension* of $f \circ \pi_S$ at the point m.

Definition 6.3. Let P be a smooth manifold, $A \subset \mathrm{Diff}_{\mathrm{L}}(P)$ a pseudogroup of local diffeomorphisms of P, and S a subset of P endowed with a topology $\mathcal{T}$ that is stronger than the relative topology. We say that the presheaf W^∞_{S/A_S} has the (A, A_S)-*local extension property* when A_S is a subpseudogroup of A and for any $f \in W^\infty_{S/A_S}(V)$ and $m \in \pi_S^{-1}(V)$ there exists an open A-invariant neighborhood U_m of m in M and $F \in C^\infty_P(U_m)^A$ such that

$$f \circ \pi_S|_{\pi_S^{-1}(V)\cap U_m} = F|_{\pi_S^{-1}(V)\cap U_m}.$$

We will say that F is a A-*invariant local extension* of $f \circ \pi_S$ at m.

Definition 6.4. Let $(P, [\cdot,\cdot])$ be a smooth Leibniz manifold and $A \subset \mathcal{P}_{\mathrm{L}}(P)$ a pseudogroup of local diffeomorphisms of P such that the sheaf of A-invariant functions on P is closed under the Leibniz bracket $[\cdot,\cdot]$. Let $S \subset P$ be a subset of P such that W^∞_{S/A_S} has the (A, A_S)-local extension property. We say that $(P, [\cdot,\cdot], A, S)$ is *Leibniz reducible* when $(S/A_S, W^\infty_{S/A_S}, [\cdot,\cdot]^{S/A_S})$ is a well defined presheaf of Leibniz algebras where for any open set $V \subset S/A_S$, the bracket $[\cdot,\cdot]^{S/A_S}_V : W^\infty_{S/A_S}(V) \times W^\infty_{S/A_S}(V) \to W^\infty_{S/A_S}(V)$ is given by

$$[f, g]^{S/A_S}_V(\pi_S(m)) = [F, G](m) \tag{6.5}$$

for any $m \in \pi_S^{-1}(V)$ and where F, G are A-invariant local extensions at m of $f \circ \pi_S$ and $g \circ \pi_S$, respectively.

The following theorem generalizes the main reduction result in [27] to the context of Leibniz manifolds with locally defined Leibniz weak symmetries.

Theorem 6.5. *Let $(P, [\cdot, \cdot])$ be a smooth Leibniz manifold and $A \subset \mathcal{P}_{\mathrm{L}}(P)$ a pseudogroup of local diffeomorphisms of P such that the sheaf of A-invariant functions on P is closed under the Leibniz bracket $[\cdot, \cdot]$. Let $S \subset P$ be a subset of P such that W^{∞}_{S/A_S} has the (A, A_S)-local extension property. Let $B^{\sharp}_{\mathrm{L}}, B^{\sharp}_{\mathrm{R}} : T^*M \to TM$ be the left and right bundle maps, respectively, associated to the Leibniz tensor of $(P, [\cdot, \cdot])$. Then $(P, [\cdot, \cdot], A, S)$ is Leibniz reducible if and only if for any $m \in S$ we have that*

$$B^{\sharp}_{\mathrm{L}}(\Delta_m) + B^{\sharp}_{\mathrm{R}}(\Delta_m) \subset [\Delta^S_m]^{\circ}, \tag{6.6}$$

where $\Delta_m := \{\mathbf{d}F(m) | F \in C^{\infty}_P(U_m)^A$ for any open A-invariant neighborhood U_m of m in $P\}$, and where $\Delta^S_m = \{\mathbf{d}F(m) \in \Delta_m | F|_{U_m \cap V_m}$ is constant for an open A-invariant neighborhood U_m of m in P and an open A_S-invariant neighborhood V_m of m in $S\}$.

Remark 6.6. If S has the relative topology then $\Delta^S_m = \{\mathbf{d}F(m) \in \Delta_m | F|_{U_m \cap S}$ is constant$\}$ for an open A-invariant neighborhood U_m of m in P.

Remark 6.7. If A consists of local Leibniz diffeomorphisms then the condition on the sheaf of A-invariant functions on P being closed under the Leibniz bracket $[\cdot, \cdot]$ is automatically satisfied.

Lemma 6.8. *Let P be a smooth manifold, $A \subset \mathrm{Diff}_{\mathrm{L}}(P)$ a pseudogroup of local transformations of P, and $S \subset P$ a subset whose topology is stronger than the relative topology and such that A_S is a subpseudogroup of A. If $\pi_S : S \to S/A_S$ is the projection, $U \subset P$ is an open A-invariant subset of P, $F \in C^{\infty}_P(U)^A$, and $V := \pi_S(U \cap S)$ then there exists a unique function $f \in W^{\infty}_{S/A_S}(V)$ such that*

$$f \circ \pi_S|_{U \cap S} = f \circ \pi_S|_{\pi_S^{-1}(V) \cap U} = F|_{\pi_S^{-1}(V) \cap U}. \tag{6.7}$$

Proof. Since by hypothesis the topology of S is stronger than the relative topology we have that for any open A-invariant subset U of P, the intersection $U \cap S$ is an open A_S-invariant subset of S. As the projection π_S is an open map, the set $V := \pi_S(U \cap S)$ is open in S/A_S. Also, the A_S-invariance of $U \cap S$ implies that $U \cap S = \pi_S^{-1}(V)$ and hence

$$\pi_S^{-1}(V) \cap U = U \cap S \cap U = U \cap S, \tag{6.8}$$

which proves the first equality in (6.7).

Now, the invariance properties of F and S imply the existence of a unique map f defined on V such that $f \circ \pi_S|_{U \cap S} = F|_{U \cap S}$ or equivalently, by (6.8), $f \circ \pi_S|_{\pi_S^{-1}(V) \cap U} = F|_{\pi_S^{-1}(V) \cap U}$. Given that by construction $\pi_S^{-1}(V) \subset U$ then for any $m \in \pi_S^{-1}(V)$, the map f satisfies (6.4) by taking in that characterization U and F, which implies that $f \in W^{\infty}_{S/A_S}(V)$. □

Proof of Theorem 6.5. We first show that if $(P, [\cdot,\cdot], A, S)$ is Leibniz reducible then $\Delta_m^S \subset [B_{\mathrm{L}}^\sharp(\Delta_m) + B_{\mathrm{R}}^\sharp(\Delta_m)]^\circ$ for all $m \in S$. Let $\alpha_m \in \Delta_m^S$; by definition there exists an open A-invariant neighborhood U_m of m in P and a function $K \in C_P^\infty(U_m)^A$ such that $\alpha_m = \mathbf{d}K(m)$ and $K|_{V_m\cap U_m}$ is constant for an open A_S-invariant neighborhood of m in S. Notice now that by definition any element in $B_{\mathrm{L}}^\sharp(\Delta_m) + B_{\mathrm{R}}^\sharp(\Delta_m)$ can be written as $X_F^{\mathrm{L}}(m)+X_G^{\mathrm{R}}(m)$ with $F, G \in C^\infty(W_m)^A$, W_m an open A-invariant neighborhood of m in P. By Lemma 6.8 there exist functions $k \in W_{S/A_S}^\infty(\pi_S(U_m \cap S))$ and $f \in W_{S/A_S}^\infty(\pi_S(W_m \cap S))$ such that

$$k \circ \pi_S|_{U_m\cap S} = K|_{U_m\cap S}, \qquad f \circ \pi_S|_{W_m\cap S} = F|_{W_m\cap S}.$$

Hence, by the Leibniz reducibility of $(P, [\cdot,\cdot], A, S)$ we have that

$$\begin{aligned}\langle \alpha_m, X_F^{\mathrm{L}}(m) + X_G^{\mathrm{R}}(m)\rangle &= [K, G](m) - [F, K](m)\\ &= [k|_W, g|_W]_W^{S/A_S}(\pi_S(m)) - [f|_W, k|_W]_W^{S/A_S}(\pi_S(m)),\end{aligned}$$

where $W = \pi_S(U_m \cap S) \cap \pi_S(W_m \cap S)$. However, given that the function C on P that is constant and equal to $K(m)$ is also an A-invariant local extension of $k \circ \pi_S$ at m, we have that

$$[k|_W, g|_W]_W^{S/A_S}(\pi_S(m)) - [f|_W, k|_W]_W^{S/A_S}(\pi_S(m)) = [C, G] - [F, C] = 0,$$

which implies that $\langle \alpha_m, X_F^{\mathrm{L}}(m) + X_G^{\mathrm{R}}(m)\rangle = 0$. Since $X_F^{\mathrm{L}}(m) + X_G^{\mathrm{R}}(m) \in B_{\mathrm{L}}^\sharp(\Delta_m) + B_{\mathrm{R}}^\sharp(\Delta_m)$ is arbitrary we have that $\alpha_m \in [B_{\mathrm{L}}^\sharp(\Delta_m) + B_{\mathrm{R}}^\sharp(\Delta_m)]^\circ$.

Suppose now that the inclusion (6.6) holds and then we will prove the reducibility of $(P, [\cdot,\cdot], A, S)$. Let $f, g \in W_{S/A_S}^\infty(V)$ and $F, G \in C_P^\infty(U_m)^A$ be local A-invariant extensions of $f \circ \pi_S$ and $g \circ \pi_S$, respectively, at a point $m \in \pi_S^{-1}(V)$. We now show that the equality:

$$[f, g]_V^{S/A_S}(\pi_S(m)) = [F, G]_{U_m}(m) \tag{6.9}$$

provides a well defined presheaf of Leibniz algebras. The only point that requires a proof is that the expression (6.9) does not depend on the local extensions utilized in the definition. The fact that $[\cdot,\cdot]^{S/A_S}$ determines a presheaf of Leibniz algebras is inherited from the properties of the bracket $[\cdot,\cdot]$ on P. Let $G' \in C_P^\infty(U_m)^A$ be another local extension of $g \circ \pi_S$ at m. This implies that $G - G'|_{\pi_S^{-1}(V)\cap U_m} = 0$ and hence $\mathbf{d}(G - G')(m) \in \Delta_m^S \subset [B_{\mathrm{L}}^\sharp(\Delta_m) + B_{\mathrm{R}}^\sharp(\Delta_m)]^\circ$. Consequently,

$$0 = \langle \mathbf{d}(G - G')(m), X_F^{\mathrm{L}}(m)\rangle = -[F, G - G']_{U_m}(m),$$

which implies that $[F, G]_{U_m}(m) = [F, G']_{U_m}(m)$ and hence guarantees the independence of (6.9) with respect to the choice of local extension for $g\circ\pi_S$. A similar argument guarantees that this definition is also independent of the choice of extension for $f \circ \pi_S$. Therefore, the expression (6.9) defines a function $[f, g]_V^{S/A_S}$ on V that actually belongs to $W_{S/A_S}^\infty(V)$ because if F and G are local A-invariant extensions of $f \circ \pi_S$ and $g \circ \pi_S$, respectively, at any point $m \in \pi_S^{-1}(V)$ then so is the function $\{F, G\}$ with respect to $\{f, g\}_V^{S/A} \circ \pi_S$ by the hypothesis on the sheaf of A-invariant functions on P being closed under the Leibniz bracket $[\cdot,\cdot]$. □

6.2. Leibniz reduction by distributions

The Leibniz reduction theorem that we presented in Section 6.1 requires the presence of a pseudogroup of transformations defined in the entire manifold. However, sometimes one may want to reduce with respect to an invariance property defined only on a subset of the manifold in question. The study of this situation is the main goal of this section.

We start by recalling the notion of decomposed space. Let $\mathcal{Z}$ be a locally finite partition of the topological space P into smooth manifolds $S_i \subset P$, $i \in I$. We assume that the manifolds $S_i \subset P$, $i \in I$ with their manifold topology are locally closed topological subspaces of P. We say that the pair $(P, \mathcal{Z})$ is a *decomposition* of P with *pieces* in $\mathcal{Z}$ when the following condition is satisfied:

(DS) If $R, S \in \mathcal{Z}$ are such that $R \cap \bar{S} \neq \emptyset$, then $R \subset \bar{S}$. In this case we write $R \preceq S$. If, in addition, $R \neq S$ we say that R is *incident* to S or that it is a *boundary piece* of S and write $R \prec S$.

The distributions in the next definition are allowed not to have constant rank, that is, we are considering *generalized* distributions.

Definition 6.9. Let P be a differentiable manifold and $S \subset P$ a decomposed subset of P. Let $\{S_i\}_{i \in I}$ be the pieces of this decomposition. The topology of S is not necessarily the relative topology as a subset of P. We say that $D \subset TP|_S$ is a *smooth distribution on S adapted to the decomposition* $\{S_i\}_{i \in I}$, if $D \cap TS_i$ is a smooth distribution on S_i for all $i \in I$. The distribution D is said to be *integrable* if $D \cap TS_i$ is integrable for each $i \in I$.

In the situation described by Definition 6.9 and if D is integrable, the integrability of the distributions $D_{S_i} := D \cap TS_i$ on S_i allows us to partition each S_i into the corresponding maximal integral manifolds. Thus, there is an equivalence relation on S_i whose equivalence classes are precisely these maximal integral manifolds. Doing this on each S_i, we obtain an equivalence relation D_S on the whole set S by taking the union of the different equivalence classes corresponding to all the D_{S_i}. We define the quotient space S/D_S as

$$S/D_S := \bigcup_{i \in I} S_i/D_{S_i}.$$

We will denote by $\pi_{D_S} : S \to S/D_S$ the natural projection.

Definition 6.10. Let $(P, [\cdot, \cdot])$ be a Leibniz manifold and $D \subset TP$ a smooth distribution on P. The distribution D is called *Leibniz* or *canonical*, if the condition $\mathbf{d}f|_D = \mathbf{d}g|_D = 0$ for any $f, g \in C^\infty_P(U)$ and any open subset $U \subset P$, implies that $\mathbf{d}[f, g]|_D = 0$.

The presheaf of smooth functions on S/D_S. In this section, we will be considering a presheaf of smooth functions on S/D_S that require less invariance properties in their definition than those that appeared in the context of quotients by pseudogroups of transformations. We define the presheaf of smooth functions C^∞_{S/D_S} on S/D_S as the map that associates to any open subset V of S/D_S the set of functions $C^\infty_{S/D_S}(V)$ characterized by the following property: $f \in C^\infty_{S/D_S}(V)$ if and only if for any $z \in V$ there exists $m \in \pi^{-1}_{D_S}(V)$, U_m open neighborhood of m in P, and $F \in C^\infty_P(U_m)$ such that

$$f \circ \pi_{D_S}|_{\pi^{-1}_{D_S}(V)\cap U_m} = F|_{\pi^{-1}_{D_S}(V)\cap U_m}. \tag{6.10}$$

We say that F is a *local extension* of $f \circ \pi_{D_S}$ at the point $m \in \pi^{-1}_{D_S}(V)$. It can be proved (see [28]) that if S is a smooth embedded submanifold of P and D_S is a smooth, integrable, and regular distribution on S then the presheaf C^∞_{S/D_S} coincides with the presheaf of smooth functions on S/D_S when considered as a regular quotient manifold.

We say that the presheaf C^∞_{S/D_S} has the (D, D_S)-*local extension property* when the topology of S is stronger than the relative topology and, at the same time, the local extensions of $f \circ \pi_{D_S}$ defined in (6.10) can always be chosen so that

$$\mathbf{d}F(n)|_{D(n)} = 0 \quad \text{for any } \; n \in \pi^{-1}_{D_S}(V) \cap U_m.$$

We say that F is a *local D-invariant extension* of $f \circ \pi_{D_S}$ at the point $m \in \pi^{-1}_{D_S}(V)$.

Definition 6.11. Let $(P, [\cdot, \cdot])$ be a Leibniz manifold, S a decomposed subset of P, and $D \subset TP|_S$ a Leibniz integrable generalized distribution adapted to the decomposition of S. Assume that C^∞_{S/D_S} has the (D, D_S)-local extension property. We say that $(P, [\cdot, \cdot], D, S)$ is *Leibniz reducible* when $(S/D_S, C^\infty_{S/D_S}, [\cdot, \cdot]^{S/D_S})$ is a well defined presheaf of Leibniz algebras where, for any open set $V \subset S/D_S$, the bracket $[\cdot, \cdot]^{S/D_S}_V : C^\infty_{S/D_S}(V) \times C^\infty_{S/D_S}(V) \to C^\infty_{S/D_S}(V)$ is given by

$$[f, g]^{S/D_S}_V(\pi_{D_S}(m)) := [F, G](m)$$

for any $m \in \pi^{-1}_{D_S}(V)$. The maps F, G are local D-invariant extensions at m of $f \circ \pi_{D_S}$ and $g \circ \pi_{D_S}$, respectively.

The proof of the following theorem mimics the corresponding implication in Theorem 6.5.

Theorem 6.12. *Let $(P, [\cdot, \cdot])$ be a Leibniz manifold with associated Leibniz tensor B, S a decomposed space, and $D \subset TP|_S$ a Leibniz integrable generalized distribution adapted to the decomposition of S. Assume that C^∞_{S/D_S} has the (D, D_S)-local extension property. Then $(P, [\cdot, \cdot], D, S)$ is Leibniz reducible if for any $m \in S$*

$$B^\sharp_{\mathrm{L}}(\Delta_m) + B^\sharp_{\mathrm{R}}(\Delta_m) \subset [\Delta^S_m]^\circ, \tag{6.11}$$

where $\Delta_m := \{\mathbf{d}F(m) | F \in C^\infty_P(U_m), \mathbf{d}F(z)|_{D(z)} = 0 \text{ for all } z \in U_m \cap S, \text{ and for any open neighborhood } U_m \text{ of } m \text{ in } P\}$ and $\Delta^S_m := \{\mathbf{d}F(m) \in \Delta_m | F|_{U_m \cap V_m} \text{ is constant for an open neighborhood } U_m \text{ of } m \text{ in } P \text{ and an open neighborhood } V_m \text{ of } m \text{ in } S\}$.

Remark 6.13. If S is endowed with the relative topology then

$$\Delta^S_m := \{\mathbf{d}F(m) \in \Delta_m | F|_{U_m \cap V_m} \;\text{ is constant for an open neighborhood } U_m \text{ of } m \text{ in } P\}.$$

Remark 6.14. As opposed to the situation in Theorem 6.5, the condition (6.11) is sufficient for Leibniz reducibility but in general not necessary. The reason behind this circumstance is that the functions that define the spaces Δ_m and Δ_m^S are not defined on saturated open sets which prevents the formulation of a result similar to Lemma 6.8. As we will see in Theorem 6.15, an alternative hypothesis that makes this condition necessary and sufficient is, roughly speaking, the regularity of the distribution $D_S := D \cap TS$.

Reduction by regular canonical distributions. Let $(P, [\cdot, \cdot])$ be a Leibniz manifold and S an embedded submanifold of P. Let $D \subset TP|_S$ be a subbundle of the tangent bundle of P restricted to S such that $D_S := D \cap TS$ is a smooth, integrable, and regular distribution on S and D is Leibniz. Our next theorem is a generalization of the main result of [24] to the context of Leibniz manifolds.

Theorem 6.15. *Let $(P, [\cdot, \cdot])$ be a Leibniz manifold with associated Leibniz tensor B and S an embedded smooth submanifold of P. Let $D \subset TP|_S$ be a canonical subbundle of the tangent bundle of P restricted to S such that $D_S := D \cap TS$ is a smooth, integrable, and regular distribution on S. Then $(P, [\cdot, \cdot], D, S)$ is Leibniz reducible if and only if*

$$B^\sharp_{\mathrm{L}}(D^\circ) + B^\sharp_{\mathrm{R}}(D^\circ) \subset TS + D. \tag{6.12}$$

Proof. We first prove that the condition (6.12) implies the Leibniz reducibility of $(P, [\cdot, \cdot], D, S)$. This implication can be obtained as a corollary of Theorem 6.12. Indeed, a result whose proof can be found as Lemma 10.4.14 in [28] guarantees that the hypotheses on D_S imply that the presheaf C^∞_{S/D_S} has the (D, D_S)-local extension property. Hence, it suffices to show that in this situation:

$$\Delta_m = D(m)^\circ, \tag{6.13}$$

$$[\Delta_m^S]^\circ = T_m S + D(m). \tag{6.14}$$

In order to prove (6.13) notice first that by definition $\Delta_m \subset D(m)^\circ$. To prove the converse inclusion take $\alpha_m \in D(m)^\circ$ arbitrary and let U_m be a submanifold chart of S around m that we can think of as $U \times V \subset \mathbf{F}_1 \oplus \mathbf{F}_2$, where U and V are open neighborhoods of the origin in two vector spaces $\mathbf{F}_1$ and $\mathbf{F}_2$, respectively. This chart can be constructed so that $m \equiv (0, 0)$ and $U_m \cap S = U$. Additionally, we can locally take $D = U \times \mathbf{E}$, with $\mathbf{E}$ a vector subspace of $\mathbf{F}_1 \oplus \mathbf{F}_2$, $TU_m = U \times V \times (\mathbf{F}_1 \oplus \mathbf{F}_2)$, and $T^*U_m = U \times V \times (\mathbf{F}_1 \oplus \mathbf{F}_2)^*$. In these coordinates $\alpha_m \equiv (0, 0, \alpha)$, with $\alpha \in \mathbf{E}^\circ$. Define $F : U \times V \to \mathbb{R}$ by $F(u, v) = \langle \alpha, (u, v)\rangle$. Note that $\mathbf{d}F(m) \equiv \mathbf{d}F(0, 0) = \alpha \equiv \alpha_m$ and that for any $(u, 0, w) \in D(u, 0)$, $u \in U$, $w \in \mathbf{E}$ we have that $\mathbf{d}F(u, 0) \cdot w = \langle \alpha, w\rangle = 0$, which implies that $\mathbf{d}F(z)|_{D(z)} = 0$ for any $z \in U_m \cap S$, as required.

We now prove (6.14). By definition $T_m S + D(m) \subset [\Delta_m^S]^\circ$. Conversely, the inclusion $[\Delta_m^S]^\circ \subset T_m S + D(m)$ holds if and only if $D(m)^\circ \cap T_m S^\circ \subset \Delta_m^S$ which, by (6.13), amounts to $\Delta_m \cap T_m S^\circ \subset \Delta_m^S$. We prove this inclusion by using again the same adapted local submanifold coordinates around the point m. Let $\alpha_m \in \Delta_m \cap T_m S^\circ$ arbitrary. As we saw above, there exists $\alpha \in \mathbf{E}^\circ$ such that $\alpha_m = \mathbf{d}F(0, 0)$, with $F : U \times V \to \mathbb{R}$ given by $F(u, v) := \langle \alpha, (u, v)\rangle$, $(u, v) \in U \times V$. Since $\alpha_m \in (T_m S)^\circ$ we have that $\mathbf{d}F(0, 0) \cdot (u, 0) = \langle \alpha, (u, 0)\rangle = 0$ for any $u \in \mathbf{F}_1$. This equality implies that $F(u, v) = \langle \alpha, (u, 0)\rangle + \langle \alpha, (0, v)\rangle = \langle \alpha, (0, v)\rangle$

for any $(u, v) \in U \times V$, and hence F is constant on $U = U_m \cap S$ which shows that $\alpha_m \in \Delta_m^S$, as required.

We now show that the reducibility of $(P, [\cdot, \cdot], D, S)$ implies (6.12) or, equivalently, that for any $m \in S$:

$$T_m S^\circ \cap D(m)^\circ \subset ((B_{\mathrm{L}}^\sharp(m) + B_{\mathrm{R}}^\sharp(m))(D(m)^\circ))^\circ. \tag{6.15}$$

We proceed again by using the same local coordinates. In this occasion we consider a non-degenerate inner product $\langle\cdot, \cdot\rangle_{\mathbf{F}_1 \oplus \mathbf{F}_2}$ on $\mathbf{F}_1 \oplus \mathbf{F}_2$ defined by $\langle(u_1, u_2), (v_1, v_2)\rangle_{\mathbf{F}_1 \oplus \mathbf{F}_2} = \langle u_1, v_1\rangle_{\mathbf{F}_1} + \langle u_2, v_2\rangle_{\mathbf{F}_2}$, with $\langle\cdot, \cdot\rangle_{\mathbf{F}_1}$ and $\langle\cdot, \cdot\rangle_{\mathbf{F}_2}$ non-degenerate inner products in $\mathbf{F}_1$ and $\mathbf{F}_2$, respectively, and $u_1, v_1 \in \mathbf{F}_1$, $u_2, v_2 \in \mathbf{F}_2$. Given that $U_m \cap S = U \subset \mathbf{F}_1$ any element $\alpha_m \in T_m S^\circ$ can be written as $\alpha_m = \langle(0, u_0), \cdot\rangle_{\mathbf{F}_1 \oplus \mathbf{F}_2}$, for some $u_0 \in \mathbf{F}_2$ or, analogously, as $\alpha_m = \mathbf{d}K(m)$, with $K \in C^\infty(U_m)$ defined by

$$K(u, v) := \langle(0, u_0), (u, v)\rangle_{\mathbf{F}_1 \oplus \mathbf{F}_2} = \langle u_0, v\rangle_{\mathbf{F}_2}. \tag{6.16}$$

Moreover, if $\alpha_m \in T_m S^\circ \cap D(m)^\circ$ then as D in these coordinates looks like $D = U \times \mathbf{E}$ for some vector subspace $\mathbf{E} \subset \mathbf{F}_1 \oplus \mathbf{F}_2$, we have that the function K defined in (6.16) is such that

$$K|_{U_m \cap S} = 0 \quad \text{and} \quad \mathbf{d}K(z)|_{D(z)} = 0 \quad \text{for any} \quad z \in U_m \cap S.$$

We have thus proven that any $\alpha_m \in T_m S^\circ \cap D(m)^\circ$ can be written as $\alpha_m = \mathbf{d}K(m)$ with K a local D-invariant extension of the zero function in S at the point $m \in S$.

Let now $\beta_m \in D(m)^\circ$. Due to the non-degeneracy of the inner product $\langle\cdot, \cdot\rangle_{\mathbf{F}_1 \oplus \mathbf{F}_2}$ there exists $w_0 \in \mathbf{F}_1 \oplus \mathbf{F}_2$ such that $\beta_m = \mathbf{d}F(m)$ with $F(u) := \langle w_0, u\rangle_{\mathbf{F}_1 \oplus \mathbf{F}_2}$, $u \in U_m$, and such that $\langle w_0, w\rangle_{\mathbf{F}_1 \oplus \mathbf{F}_2} = 0$ for any $w \in \mathbf{E}$. The regularity of the distribution D_S implies via a result of Godement (see Lemma 3.5.26 in [2]) that the neighborhood U_m can be shrunk so that there exists a smooth submanifold T of $U_m \cap S$ (called a *local slice* of D_S) and a smooth map $s : U_m \cap S \to T$ such that $s|_T$ is the identity map on T and the integral leaf $\mathcal{L}_u$ of D_S that contains any arbitrary point $u \in U_m \cap S$ is such that $\mathcal{L}_u \cap T = \{s(u)\}$. Notice now that since $\mathbf{d}F|_{U_m \cap S} \cdot D_S|_{U_m \cap S} = 0$, we can use the slice and the map $F|_{U_m \cap S}$ to define another map $f \in C^\infty_{S/D_S}(\pi_{D_S}(U_m \cap S))$ as the unique map that satisfies

$$f \circ \pi_{D_S}(z) = F(z) = F(s(z)) \quad \text{for any} \quad z \in U_m \cap S.$$

Using the constructions in the last two paragraphs we can now write for any $\alpha_m \in T_m S^\circ \cap D(m)^\circ$ and any $\beta_m^1, \beta_m^2 \in D(m)^\circ$:

$$\begin{aligned} &\langle \alpha_m, B_{\mathrm{L}}^\sharp(m)(\beta_m^1) + B_{\mathrm{R}}^\sharp(m)(\beta_m^2)\rangle \\ &\quad = [K, F](m) - [G, K](m) = [0, f]^{S/D_S}_{\pi_{D_S}(U_m \cap S)}(\pi_{D_S}(m)) - [g, 0]^{S/D_S}_{\pi_{D_S}(U_m \cap S)}(\pi_{D_S}(m)) \\ &\quad = [0, F](m) - [G, 0](m) = 0, \end{aligned} \tag{6.17}$$

where in the last equality we used the Leibniz reducibility of $(P, [\cdot, \cdot], D, S)$ to write

$$[0, f]^{S/D_S}_{\pi_{D_S}(U_m \cap S)}(\pi_{D_S}(m)) = [0, F](m) \quad \text{and} \quad [g, 0]^{S/D_S}_{\pi_{D_S}(U_m \cap S)}(\pi_{D_S}(m)) = [G, 0](m),$$

since the zero function on M is also an extension of the zero function on S that can be used instead of K in the definition of the bracket. The expression (6.17) establishes (6.15). □

Example 6.16. We now illustrate the use of Theorem 6.15 by reducing the Leibniz system that we constructed in (iv) of Example 5.9 to encode the dynamics of the non-holonomically constrained particle. We recall that in this situation the Leibniz manifold consists of the pair $(T^*\mathbb{R}^3, [\cdot,\cdot])$, where $[\cdot,\cdot]$ is the Leibniz bracket induced by the tensor that in coordinates (x, y, z, p_x, p_y, p_z) is given by the expression:

$$\tilde{B}^\sharp(x, y, z, p_x, p_y, p_z) = \begin{pmatrix} 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ \dfrac{-y^2}{1+y^2} & 0 & \dfrac{y}{1+y^2} & 0 & \dfrac{-p_z}{1+y^2} & 0 \\ 0 & -1 & 0 & 0 & 0 & 0 \\ \dfrac{y}{1+y^2} & 0 & \dfrac{-1}{1+y^2} & 0 & \dfrac{-yp_z}{1+y^2} & 0 \end{pmatrix}. \tag{6.18}$$

The equations of motion of the non-holonomically constrained particle given in (5.5) coincide with the Leibniz vector field associated to the Hamiltonian function $H(x, y, z, p_x, p_y, p_z) = (1/2)(p_x^2 + p_y^2 + p_z^2)$.

A straightforward verification shows that the embedded submanifold:

$$S := \{(x, y, z, a, p_y, 0) | x, y, z, p_y \in \mathbb{R}\}$$

is left invariant by the dynamics of the system, where $a \in \mathbb{R}$ is the constant that appears in the definition of the constraint (5.5).

Consider now the distribution $D \subset T(T^*\mathbb{R}^3)|_S$ given by the vectors of the form:

$$D(m) := \{(u, 0, v, 0, 0, 0) | u, v \in \mathbb{R}\}$$

for any $m \in S$. It is easy to see that D is a canonical subbundle of the tangent bundle of $T^*\mathbb{R}^3$ restricted to S such that $D_S := D \cap TS$ is a smooth, integrable, and regular distribution on S. We are now going to show that condition (6.12) in Theorem 6.15 is satisfied and hence that the Leibniz structure $(T^*\mathbb{R}^3, [\cdot,\cdot])$ induces a reduced Leibniz system $(S/D_S, [\cdot,\cdot]^{S/D_S})$ via the prescription in Definition 6.11. Indeed, in this case

$$\tilde{B}^\sharp_{\mathrm{L}}(m)(D(m)^\circ) + \tilde{B}^\sharp_{\mathrm{R}}(m)(D(m)^\circ) = \{(q, r, s, 0, t, 0) | q, r, s, t \in \mathbb{R}\}$$

for any $m \in S$, which is equal to $T_m S + D(m) = T_m S$. The reduced space S/D_S can be identified with the Euclidean space $\mathbb{R}^2$ endowed with the skew symmetric Leibniz tensor B_{S/D_S} given by the matrix:

$$B^\sharp_{S/D_S} = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}.$$

The dynamics of the non-holonomically constrained particle drops to the reduced space in the form of a standard Hamiltonian dynamical system with Hamiltonian function $\bar{h}(u, v) = (1/2)v^2$.

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